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2003 J. Phys. A: Math. Gen. 36 11881

(http://iopscience.iop.org/0305-4470/36/47/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 11881–11899 PII: S0305-4470(03)66032-4

An eigenvalue problem related to the non-linear σ-model: analytical and numerical results

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Received 9 July 2003
Published 12 November 2003
Online at stacks.iop.org/JPhysA/36/11881

Abstract
An eigenvalue problem relevant for the non-linear sigma model with singular metric is considered. We prove the existence of a non-degenerate pure point spectrum for all finite values of the size R of the system. In the infrared (IR) regime (large R) the eigenvalues admit a power series expansion around the IR critical point $R \to \infty$. We compute high order coefficients and prove that the series converges for all finite values of R. In the ultraviolet (UV) limit the spectrum condenses into a continuum spectrum with a set of residual bound states. The spectrum agrees nicely with the central charge computed by the thermodynamic Bethe ansatz method.

PACS number: 03.70.+k

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The non-linear sigma models in two-dimensional (2D) spacetime are widely used in field theory as continuous models of two-dimensional spin systems (see, e.g., [1–4]) as well as in relation to string theory (e.g., [5–8]). The general 2D sigma model (SM) is defined through the action

$$\mathcal{A}[G] = \frac{1}{2}\int G_{ij}(X)\partial_\mu X^i \partial_\mu X^j \, \mathrm{d}^2 x \qquad (1)$$

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0305-4470/03/4711881+19$30.00

where the coordinates x^μ, $\mu = 1, 2$ span a 2D flat spacetime, while the fields X^i, $i = 1, \dots, d$ are coordinates in a d-dimensional Riemannian manifold called target space. The symmetric matrix G_{ij} is the corresponding metric tensor.

The standard approach to 2D sigma models is perturbation theory. If the curvature of G_{ij} is small, one can use the following one loop renormalization group equation from [3]:

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{ij} = -\frac{1}{2\pi} R_{ij} \tag{2}$$

where t is the RG 'time' (the logarithm of scale) and R_{ij} is the Ricci tensor of G.

The simplest examples of 2D sigma models are SM with two-dimensional target space ($d = 2$). In this case $R_{ij} = \mathcal{R}\delta_{ij}$ where $\mathcal{R}$ is the scalar curvature. Then we can always choose (at least locally) conformal coordinates for which

$$G_{ij} = \mathrm{e}^{\Phi}\delta_{ij} \tag{3}$$

with a single function Φ.

An important role in the analysis of 2D sigma models is played by the effective central charge $c(R)$. This dimensionless function contains the information about the UV and IR properties of the theory and it is related to the ground state energy $E_0(R)$ of the corresponding quantum system, living on a finite space circle of length R:

$$E_0(R) = -\frac{\pi c(R)}{6R}. \tag{4}$$

For SM with two-dimensional target space $c_{\mathrm{UV}} = c(0) = 2$.

In integrable theories this quantity can be calculated exactly using thermodynamic Bethe ansatz (TBA) equations [9, 10]. This problem is, however, much more complicated for the excited levels $E_i(R)$, so it is useful to have some independent approach for their calculation. It was shown in [11] that for the sigma models with $d = 2$, in the one-loop approximation (equation (2)), this problem can be reduced to the eigenvalue problem for the operator:

$$\hat{h} = -\frac{1}{2}\nabla_t^2 + \frac{1}{8}\mathcal{R}_t \qquad \hat{h}\Psi_i = \frac{\pi e_i(R)}{6}\Psi_i. \tag{5}$$

Here $\nabla_t^2 = \mathrm{e}^{-\Phi}\left(\partial_x^2 + \partial_y^2\right)$ is the Laplace operator and $\mathcal{R}_t$ is the scalar curvature in the SM metric renormalized at the scale R:

$$t - t_0 = \log R\Lambda_0 \tag{6}$$

where Λ_0 is the normalization parameter. This operator is self-conjugate with respect to the scalar product in the SM metric:

$$(\Psi_1, \Psi_2) = \int \Psi_1^* \Psi_2 \,\mathrm{e}^{\Phi}\,\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

The effective central charge $c(R)$ in the one-loop approximation can be expressed through the lowest eigenvalue:

$$c(R) = 2 - e_0(R) \tag{8}$$

and the excited levels $E_i(R) = E_0(R) + \pi(e_i(R) - e_0(R))/6R$. We note that if in the IR limit SM flows to the critical point described by conformal field theory (CFT) then the numbers $\Delta_i = (e_i(\infty) - e_0(\infty))/24$ coincide with conformal dimensions of the fields in this CFT.

The eigenvalue problem (equation (5)) with the natural scalar product provided by the metric $G_{ij}(X)$ can be applied to the analysis of 2D sigma models with target space of arbitrary dimension d . It follows from Zamolodchikov's c-theorem that the effective central charge defined in [12] through the correlation functions of the energy momentum tensor is non-increasing as a function of the scale R. The effective central charge defined by equation (8) (with $d - e_0(R)$ in the rhs) also satisfies this remarkable property, which follows from one of the results of [13], where it was shown that the lowest eigenvalue of the operator $\hat{h}$ is a non-decreasing function of R.

A well-known solution to equation (2) (see [11]) defines the axially symmetric metric of the 'sausage' SM (an integrable deformation of the $O(3)$ non-linear sigma model), which is described by the action:

$$\mathcal{A}_{\mathrm{ssg}} = \int \frac{(\partial_\mu X)^2 + (\partial_\mu Y)^2}{a(t) + b(t)\cosh 2Y}\,\mathrm{d}^2x \tag{9}$$

where $a(t) = \nu \coth 2u$, $b(t) = \nu/\sinh 2u$ and $u = \nu(t_0 - t)/4\pi$. It is easy to see from the explicit form of metric (9) that operator $\hat{h}/\nu$ depends only on the variable u and does not depend on parameter ν. It means that

$$e_i^{\mathrm{ssg}}(R) = \frac{\nu}{4\pi}\kappa_i^{\mathrm{ssg}}(u). \tag{10}$$

where after the substitution $\Psi = \exp(\mathrm{i}mx)\Psi_m(y)$ $(m \in \mathbb{Z})$ the scaling function $\kappa_i^{\mathrm{ssg}}(u)$ is the eigenvalue of the Sturm–Liouville problem:

$$\left[-\partial_y^2 + m^2 + \frac{1 + \cosh 2u \cosh 2y}{(\cosh 2u + \cosh 2y)^2} - \frac{\frac{1}{6}\kappa_i^{\mathrm{ssg}}(u)\sinh u}{\cosh 2u + \cosh 2y}\right]\Psi_m^{(i)} = 0 \tag{11}$$

where the eigenfunctions $\Psi_m^{(i)}$ have finite norm according to equation (7). For the ground state $\kappa_0^{\mathrm{ssg}}(u)$ this problem was studied in [11].

In this paper we consider the eigenvalue problem for the sigma model which corresponds to another solution of RG equation (2). This solution can be obtained by analytic continuation $Y \to Y + \mathrm{i}\pi/4$, $u \to u + \mathrm{i}\pi/4$ from the solution for the sausage model. The corresponding action can be written as

$$\mathcal{A} = \int \frac{(\partial_\mu X)^2 + (\partial_\mu Y)^2}{\alpha(t) + \beta(t)\sinh 2Y}\,\mathrm{d}^2x \tag{12}$$

where $\alpha(t) = \nu \tanh 2u$, $\beta(t) = \nu/\cosh 2u$ and $u = \nu(t_0 - t)/4\pi$. This metric has a singularity at $Y = -u$. It means that the coordinate Y in target space should be considered only in the region $Y > -u$. The curvature $\mathcal{R}$ also has a singularity at this point. However, for small values of parameter ν the curvature is not small only in the narrow region ($\delta Y \sim \nu$) in the vicinity of the singularity. A more careful analysis shows that the one loop approximation is valid for the calculation of the observables in SM (12). The relative correction to the one-loop approximation as well as in the sausage SM has the order $\nu \log(1/\nu)$. The eigenvalue equation for the scaling functions $\kappa_i(u)$ $(e_i(R) = \nu\kappa_i(u)/4\pi)$ now has the form:

$$\left[-\partial_y^2 + m^2 - \frac{1 - \sinh 2u \sinh 2y}{(\sinh 2u + \sinh 2y)^2} - \frac{\frac{1}{6}\kappa_i(u)\cosh u}{\sinh 2u + \sinh 2y}\right]\Psi_m^{(i)} = 0. \tag{13}$$

The solution $\Psi_m^{(i)}(y)$ should now satisfy the boundary condition

$$\Psi_m^{(i)}(y) \underset{y\to -u}{\approx} (y + u)^{1/2} \tag{14}$$

and it must be square integrable with respect to the natural norm

$$\|\Psi_m^{(i)}\|^2 = \int_{-u}^{\infty} \frac{|\Psi_m^{(i)}(y)|^2}{\sinh 2u + \sinh 2y}\,\mathrm{d}y. \tag{15}$$

In the IR limit $u \to -\infty$ the metric (12) has an asymptotic which can be written in the form of equation (3) with $\exp\{-\Phi_{\mathrm{IR}}\} = \frac{1}{2}\nu(\exp\{2Z\} - 1)$, with $Z = Y + u$. For discrete values of the parameter $\nu = 4\pi/N$ the SM with this metric can be derived from the $SU(2)$ level N WZW models by gauging $U(1)$ symmetry (see [14, 15] for details). The resulting $SU(2)_N/U(1)$ coset model described by the SM with metric $\exp(\Phi_{\mathrm{IR}})\delta_{ij}$ coincides with Z_N parafermionic CFT of [16]. For general values of u the quantum field theory (QFT) corresponding to SM (12) can be considered as the deformation of parafermionic CFT. It is natural to expect that it will be a massless theory describing the RG flow from rather non-trivial UV field theory (which is also well defined for the same discrete values of parameter ν) with $c_{\mathrm{UV}} = 2$ to the parafermionic CFT with $c_{\mathrm{IR}} = 2 - 6/(N+2)$ in the IR limit. The scaling functions $\kappa_i(u)$ in this case describe the RG dynamics of energy levels from the UV regime to IR asymptotics, where they define (with relative accuracy $O(1/N)$) the spectrum of anomalous dimensions of parafermionic CFT (see appendix C).

This massless QFT is integrable and can be described in terms of factorized scattering theory for massless excitations. For $\nu = 4\pi/N$ with $N \geqslant 3$ the ground state energy equation (4) of the SM (12) (as well as that of the sausage SM (9), of [11]) admits an exact calculation by the TBA method. These equations will be described in section 5.

Both eigenvalue problems, equations (1) and (13), are believed to have a purely discrete spectrum. The ground state eigenvalue κ_0^{ssg} of equation (11) was studied in [11], where the asymptotics of this function in the regimes $u \to 0$ and $u \to \infty$ were found. This function was studied numerically in [17] and the result was in perfect agreement with the scaling function calculated from TBA equations (see section 5). Actually it is rather easy today to solve the spectral problem, equation (11), by using sparse matrix techniques available in mathematical libraries. Good accuracy can be achieved by introducing a multi-grid discretization and using Richardson extrapolation. This approach, however, is not immediately applicable to the new equation (13), due to its singular nature. In the following we shall bring the equation to a form which is suitable for a detailed perturbative analysis (section 2), to a second form which allows an accurate asymptotic analysis in the UV regime (Lamé form, section 3) as well to a third form, more suitable for a purely numerical approach (section 4). Finally (section 5) we shall exhibit the matching of the ground state of equation (13) with the central charge of the modified TBA system whose structure is given in figure 8. The interested reader will find some further mathematical details in the appendices.

2. The connection with Heun's equation

To bring equation (13) to a more manageable form, we begin by re-absorbing the integration measure into the wavefunction. By defining

$$\begin{aligned}\Psi(y) &= \sqrt{\rho(y)}\phi(y)\\ \rho(y) &= \sinh 2y + \sinh 2u\end{aligned}\tag{16}$$

we find

$$-\frac{\mathrm{d}}{\mathrm{d}y}\left(\rho(y)\frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\right) + (m^2\rho(y) - \sinh 2y)\phi(y) = \frac{1}{6}\kappa\cosh 2u\,\phi(y).\tag{17}$$

Putting $x = \mathrm{e}^{-2(y+u)}$ and $w = -\mathrm{e}^{-4u}$, equation (17) is transformed into the following:

$$\phi''(x) + \left(\frac{1}{x-1} + \frac{1}{x-w}\right)\phi'(x) + \frac{(1-m^2)x^2 - \frac{1}{6}\kappa(1-w)x + m^2 + w}{4x^2(x-1)(x-w)}\phi(x) = 0\tag{18}$$

of Fuchsian type. A further substitution $\phi \to \sqrt{x}f(x)$ reduces equation (18) to the form

$$f''(x) + \left(\frac{m+1}{x} + \frac{1}{x-1} + \frac{1}{x-w}\right)f'(x) + \frac{(1+m)x - \mathfrak{q}}{x(x-1)(x-w)}f(x) = 0\tag{19}$$

where the so-called *accessory parameter* $\mathfrak{q}$ is given by

$$\mathfrak{q} = \tfrac{1}{24}(\kappa(1 - w) + 6(1 + w)(1 + 2m)). \tag{20}$$

This equation was analysed by Heun in 1888 (see [18, 19]) who considered a general linear differential equation of second order with four Fuchsian singularities. In Heun's notation, the solution is formally given by $F(w; \mathfrak{q}; 1 + m, 1, 1 + m, 1; x)$, but this is of little use in practice. We gain some insight from the fact that the limit $w \to -\infty$ is a case of *confluence* of singularities which takes us back to the hypergeometric equation (see [19] for a general treatment).

2.1. The eigenvalue problem in algebraic form

It is well known that series solutions for Heun's equations can be most conveniently constructed using the basis of Jacobi polynomials $P_n^{(m,0)}(1 - 2x)$ (see, e.g., [20], vol III). We are now going to show how to solve the eigenvalue problem by exploiting this favourable basis: the problem will reduce to finding the spectrum of an (infinite-dimensional) tridiagonal matrix for which efficient algorithms are well-known to exist [21]. Let us consider equation (19): after setting $y = 1 - 2x$, we can easily expand the solution in a series of Jacobi polynomials $\mathcal{P}_n^m \equiv P_n^{(m,0)}(y)$ by converting the differential equation in the form

$$\mathcal{H} f \equiv \left\{(1 - 2w - y)\widehat{N}(\widehat{N} + m + 1) + (1 - y^2)\frac{\mathrm{d}}{\mathrm{d}y} - (1 + m)y\right\} f = (2\mathfrak{q} - m - 1) f \tag{21}$$

where $\widehat{N}\mathcal{P}_n^m = n\mathcal{P}_n^m$. Now we can use the basic properties of Jacobi's polynomials (see, e.g., [20], vol II, [22]) to reduce the operator $\mathcal{H}$ to the form $\mathcal{H}f = (1 - 2w)\mathcal{H}_0 f + \mathcal{V} f$ whose action on the basis vectors is particularly simple:

$$\begin{aligned}
\mathcal{H}_0\mathcal{P}_n^m &= n(n + m + 1)\mathcal{P}_n^m \\
\mathcal{V}\mathcal{P}_n^m &= m\frac{(2 + m)n^2 + (m + 1)(m + 2)n + m(m + 1)}{(m + 2n)(m + 2n + 2)}\mathcal{P}_n^m \\
&\quad - 2\frac{n^2(m + n)^2}{(m + 2n)(m + 2n + 1)}\mathcal{P}_{n-1}^m - 2\frac{(n + 1)^2(m + n + 1)^2}{(m + 2n + 1)(m + 2n + 2)}\mathcal{P}_{n+1}^m.
\end{aligned} \tag{22}$$

We can now conveniently study the spectrum of $\mathfrak{q}$ using this tridiagonal matrix representation, by applying, for instance, the technique of *Sturm sequences and bisection* [21]. The matrix representation also lends itself to a very simple perturbation series expansion, as we discuss in the next section. We shall have to refer to the matrix representation of $\mathcal{H}$ in the orthonormal basis $\phi_n^m = (-)^n\sqrt{\frac{1}{2n+m+1}}\mathcal{P}_n^m$ as $V_{nn'}$:

$$\begin{aligned}
V_{n+1,n} &= \frac{2(n + 1)^2(n + m + 1)^2}{(2n + m + 2)\sqrt{(2n + m + 2)^2 - 1}} \\
V_{n-1,n} &= \frac{2n^2(n + m)^2}{(2n + m)\sqrt{(2n + m)^2 - 1}}
\end{aligned} \tag{23}$$

diagonal terms being unchanged, but the former $\mathcal{V}$, being rational in its indices, is more convenient for the calculation of perturbative coefficients; we show in appendix A that we can use $\mathcal{V}$ without modifying the standard algorithm.

2.2. Perturbation theory

The tridiagonal matrix representation of $\mathcal{H}$, when rewritten as $\mathcal{H} = \varepsilon^{-1}(\mathcal{H}_0 + \varepsilon\mathcal{V})$, can be used to calculate a perturbative expansion in the parameter

$$\varepsilon = (1 - 2w)^{-1} = (1 + 2\,\mathrm{e}^{-4u})^{-1}. \tag{24}$$

The convergence of the expansion is governed by the Kato–Rellich theorem[5]: let there exist constants a, b such that $\|\mathcal{V}\phi\| \leqslant a\|\phi\| + b\|\mathcal{H}_0\phi\|$ ($\|\cdot\|$ denotes L_1-norm). Then the perturbative expansion defines a regular analytic function for $|b\varepsilon| < 1$. For a tridiagonal matrix it is not so difficult to find norm estimates; in our case it is simple algebra to check that *the column sums of the matrix elements of $\mathcal{V}$ coincide with the diagonal matrix elements of $\mathcal{H}_0$, up to an additive constant*, therefore we have

$$\mathcal{V} = \mathcal{S}(\mathcal{H}_0 + (m+1)\mathbb{I}) \tag{25}$$

with $\mathcal{S}$ a stochastic matrix[6] and $\mathbb{I}$ the identity matrix. Hence it follows:

$$\|\mathcal{V}\phi\| = \|\mathcal{S}(\mathcal{H}_0 + (m+1)\mathbb{I})\phi\| \leqslant \|\mathcal{H}_0\phi\| + (m+1)\|\phi\| \tag{26}$$

which implies that the perturbative series will converge for $\varepsilon < 1$, that is *for all u*. Actually we can say more: let us denote by $R(H,\mu) = (H+\mu)^{-1}$ the resolvent operator; by applying Schur's test to the matrix $R(\mathcal{H}_0,\mu)V$ (the symmetric version of $\mathcal{V}$) one concludes that

$$\|R(\mathcal{H}_0,\mu)V\| \leqslant 1 \tag{27}$$

if $\mu \geqslant m + 13/8$. Now the resolvent of $\mathcal{H}$ satisfies the Lippman–Schwinger equation

$$R(\mathcal{H},\mu) = R(\mathcal{H}_0,\mu) - \varepsilon R(\mathcal{H}_0,\mu)VR(\mathcal{H},\mu) \tag{28}$$

which for $|\varepsilon| < 1$ can be inverted to give

$$R(\mathcal{H},\mu) = (1 + \varepsilon R(\mathcal{H}_0,\mu)V)^{-1}R(\mathcal{H}_0,\mu). \tag{29}$$

Since $R(\mathcal{H}_0,\mu)$ is a compact operator, equation (29) states that *the resolvent $R(\mathcal{H},\mu)$ is itself a compact operator*, being the product of a compact operator with a bounded one. This implies that the spectrum of $\mathcal{H}$ is purely discrete.

The expansion can now be computed rather easily by the standard recursive algorithm (see appendix A). Details on the series expansion can be found in appendix B, where we prefer to adopt a different parameter which naturally appears in the Lamé formulation of the next section, namely

$$\lambda = (1 + \mathrm{e}^{-4u})^{-1} = \frac{2\varepsilon}{1+\varepsilon} \qquad \varepsilon = \frac{\lambda}{2-\lambda}. \tag{30}$$

The expansion in powers of λ turns out to be simpler and with better convergence properties; indeed the substitution $\varepsilon \to \lambda$ is just a special case of Euler's (E, q)-method [26]. We present just a sample of the infinite number of different series expansions, since we believe that nobody would like to copy them from paper but would rather prefer getting the code which generated the expansion[7]. The first few terms for the ground state value of κ at fixed m are the following:

$$\frac{1}{6}\kappa_{m,0} = 1 + 2m - \frac{2m}{m+2}\lambda - \frac{4(m+1)^3}{(m+2)^3(m+3)}\lambda^2 - \frac{8(m+1)^3(2m^2+5m+4)}{(m+2)^5(m+3)(m+4)}\lambda^3 + O(\lambda^4) \tag{31}$$

[5] See for instance [24, 25].

[6] I.e. $\sum_i \mathcal{S}_{i,j} \equiv 1$.

[7] Matlab and Mathematica codes are available at the web site www.fis.unipr.it/∼onofri.

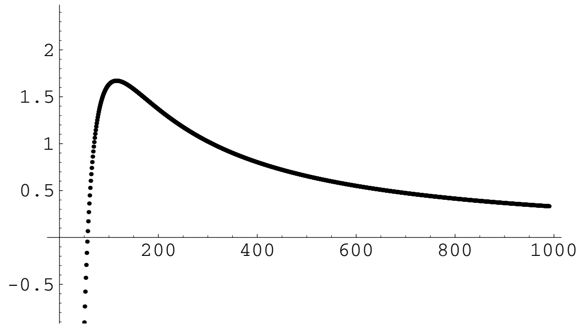

Figure 1. Relative deviation of Υ expansion coefficients from those of its leading asymptotics ($\times 10^5$) (equation (34)).

while for the excited states, after putting $j = \frac{1}{2}m + n$, we have (see also appendix C)

$$\frac{1}{6}\kappa_{m,n} = (2j+1)^2 - m^2 - \frac{[4j(j+1) - m^2]^2}{8j(j+1)}\lambda - \frac{1}{2^9(2j+1)}\left[\frac{(4(j+1)^2 - m^2)^4}{(j+1)^3(2j+3)} - \frac{(4j^2 - m^2)^4}{j^3(2j-1)}\right]\lambda^2 + O(\lambda^3). \tag{32}$$

Using floating point arithmetic we may quickly explore very high orders, with due attention to truncation errors which accumulate along the iteration. The asymptotic behaviour of the coefficients shows very clearly a limit $c_{n+1}/c_n \to 1$, confirming that the series converges in the unit circle, which means in the domain $|\mathrm{e}^{4u}/(1+\mathrm{e}^{4u})| < 1$. In the complex u plane this is a domain which includes the whole real axis. However, as we venture along the positive real axis, the convergence is critically slowed down: to go deep in the UV region we may be obliged to sum a really huge number of terms, or try some resummation, e.g., via Padé approximants. Since high order coefficients are easily computed, however, we may try to extract the asymptotic behaviour of $\kappa(u)$ for large positive u by analysing the asymptotic behaviour of the coefficients. For example, we can verify that the UV asymptotics of the ground state eigenvalue $\kappa_0(u)$ coincides with that of function $\kappa_0^{\mathrm{ssg}}(u)$ (see section 3) and has a form:

$$\kappa_0(u) = \frac{3\pi^2}{2(u + \log 4)^2} + O(1/u^5). \tag{33}$$

When expressed in terms of ε this formula can be expanded in a power series and the coefficients compared to those coming from perturbation theory. Sub-dominant terms tend to mask the simple $n^{-1}(\log n)^{-3}$ behaviour one should expect; we find that it is more accurate to compare the Taylor coefficients of the function $1/(4\mathrm{q} - 1)$, which is actually diverging at $\varepsilon \to 1$, with those of its leading term $O(\log(1-\varepsilon)^2)$. However, to make the comparison even more transparent, we may look for a special function whose behaviour at $\varepsilon \to 1$ is the simplest possible. Let us observe that $2\mathrm{q} - 1$ turns out to be an *odd* function of ε; the new function $\sqrt{\varepsilon/(4\mathrm{q} - 2 + \varepsilon)}$ has the simple $O(\log(1-\varepsilon))$ leading singular behaviour at $\varepsilon \to 1$, and it is an *even* function of ε. We argue that its leading behaviour should then be

$$\Upsilon(\varepsilon) \approx \frac{1}{2\pi\varepsilon}\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \tag{34}$$

In fact, we find that the expansion of Υ matches the perturbative series with a high accuracy (see figure 1 where the deviation is magnified 10^5 times). The asymptotic behaviour will be recovered in a very precise way numerically in section 4, hence its extraction from the perturbative series appears to be of purely academic interest. Anyhow, assuming equation (34) it follows from equation (20):

$$\kappa \equiv \frac{24\mathfrak{q} - 6(1+w)}{1-w} \approx 6(4\mathfrak{q}-1) = 6/\Upsilon^2 \approx \frac{3\pi^2}{2u^2} \tag{35}$$

since $\frac{1}{4}\log((1+\varepsilon)/(1-\varepsilon)) = \frac{1}{4}\log(1+\mathrm{e}^{4u}) \approx u$ as $\varepsilon \to 1$.

2.3. Sausage model equation

The sausage model was our starting point. Now we go back to it and show how it fits into the correspondence with Heun's equation. The scaling function is defined in equation (11). We shall now look for its algebraic equivalent as we did in section 2. Since the two equations are related by analytic continuation, it will not come as a surprise that the differential equation is *the same*, up to a map $w \to \bar{w}, \mathfrak{q} \to \bar{\mathfrak{q}}$. The range of values for the problem of SM (12) is $\frac{1}{4} < \mathfrak{q} < \frac{1}{2}, w < 0$, while the sausage is characterized by $\frac{1}{2} < \bar{\mathfrak{q}} < 1, w > 0$. The point is that equation (11) can be brought to Heun's form by the transformation $\xi = \exp\{2(y+u)\}$. The singularities are now located at $\{0, -1, -w, \infty\}$, with $w = \exp\{4u\}$ and $24\mathfrak{q} = 6(1+w) + \kappa(u)(w-1)$. The equation is actually *the same* as the one we find for the SM (12), but the domain involved is the positive real line instead of the unit interval and the singularities are differently situated. By applying well-known transformation properties of Heun's equation ([18, 23]) we can reposition the domain on the unit interval (0, 1), the singular points being now 0, 1, $\bar{w} = w/(w-1), \infty$ and the new accessory parameter is given by

$$\bar{\mathfrak{q}} = \frac{w - \mathfrak{q}}{w-1}. \tag{36}$$

Note that the map $\{\mathfrak{q}, w\} \to \{\bar{\mathfrak{q}}, \bar{w}\}$ is involutory with $w = \infty$ as the only fixed point, the interval $\frac{1}{2} < \bar{\mathfrak{q}} < 1$ being mapped onto $1 < \mathfrak{q} < \infty$.

Hence we can use the same algorithm of the previous section in a different domain. The 'magic' here is provided by the analyticity properties of the models involved. The complex shift transforming the 'sausage' model into the SM (12) does not modify very much the eigenvalue equation, which turns out to be the same equation in a different domain.

3. The Lamé formulation

The parameter λ defined by equation (30) is naturally related to a reformulation of equation (13) close to the Lamé elliptic equation. If we define the modulus of Jacobi elliptic functions

$$k^2 = \lambda = 1/(1+\exp(-4u)) \tag{37}$$

then the substitution

$$\mathrm{e}^{y-u} = \frac{\mathrm{dn}\,(z|k^2)}{k\,\mathrm{sn}(z|k^2)} \qquad \psi_m = \sqrt{\frac{\mathrm{sn}(z|k^2)\,\mathrm{dn}\,(z|k^2)}{\mathrm{cn}(z|k^2)}}\Psi_m \tag{38}$$

maps the point $y = \infty$ to $z = 0$, the point $y = -u$ to $z = K$, where $K(k^2)$ is the real period of Jacobi elliptic functions, and it turns equation (13) to the form:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\mathrm{dn}^2(2z|k^2)}{\mathrm{sn}^2(2z|k^2)} + \frac{m^2\mathrm{cn}^2(z|k^2)}{\mathrm{sn}^2(z|k^2)\,\mathrm{dn}^2(z|k^2)}\right)\psi_{m,n} = \frac{1}{6}\kappa_{m,n}\psi_{m,n} \tag{39}$$

with the boundary conditions $\psi_m \sim z^{m+1/2}$ at $z \to 0$; $\psi_m \sim (K-z)^{1/2}$ at $z \to K$. This equation can be studied analytically in two limits $k^2 \to 0$ ($u \to -\infty$) and $k^2 \to 1$ ($u \to +\infty$). In the first case we can develop the standard perturbation theory near the exact solutions $\psi_{m,n}(z) = \sqrt{\sin 2z}\cos^m(z) P_n^{(m,0)}(\cos 2z)$ where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. This perturbation theory gives the same IR expansion for the eigenvalues which was considered in previous section.

In the opposite limit $u \to \infty$, $k \to 1$ and the real period $K \sim -\frac{1}{2}\log((1-k^2)/16) \sim 2u + 2\log 2 \to \infty$. In this case the potential term in equation (39) is equal to m^2 almost everywhere and near the points $z = 0$; $z = K$ it can be approximated with exponential in u accuracy by the potentials:

$$V(z) = -\frac{1}{\sinh^2 2z} + m^2 \coth^2 z \qquad 0 < z \ll K \tag{40}$$

and

$$V_1(z_1) = -\frac{1}{\sinh^2 2z_1} + m^2 \tanh^2 z_1 \qquad 0 < z_1 \equiv K - z \ll K. \tag{41}$$

Both these potentials appeared in [27], where the spectrum of CFT describing Witten's two-dimensional Euclidean black hole [28] was studied. There, it was noted that potential $V_1(z_1)$ is attractive and has the bound state solutions:

$$\psi_{m,n} = \sqrt{\tanh z_1}(\cosh z_1)^{2n-m+1} F(-n, -n+m, m-2n; 1-\tanh^2 z_1) \tag{42}$$

where $F(a,b,c,z)$ is Gauss' hypergeometric function. These solutions are normalizable for integer $n < (m-1)/2$ and give the levels

$$\tfrac{1}{6}\kappa = \left\{m^2 - (2n+1-m)^2 | n = 0, 1, \ldots, \left[\tfrac{1}{2}m\right] - 1\right\}. \tag{43}$$

The corresponding eigenvalues of equation (39) approach these levels exponentially in u.

The states corresponding to these levels describe the discrete degrees of freedom which survive in UV asymptotics of the SM (12). We note that the UV limit of this theory is closely related to the $SL(2,R)/U(1)$ coset CFT, which was studied in detail in [27]. The bound states there correspond to the principal discrete series representations of $SL(2,R)$ and play an essential role for the string theory interpretation of the coset model. We plan to discuss the physical relevance of these discrete states for the dynamics of SM (12) in a future publication.

The potential $V(z)$ is repulsive and does not have normalizable solutions. For $2n \geqslant m-1$ we parametrize $\kappa_{m,n}/6 = m^2 + p_n^2$. Then solutions regular at $z = 0$ and at $z_1 = K - z = 0$ are found to be

$$\psi_{m,n}(z) = (\tanh z)^{m+\frac{1}{2}}(\cosh z)^{\mathrm{i}p} F\left(\tfrac{1}{2}(1+m-\mathrm{i}p), \tfrac{1}{2}(1+m-\mathrm{i}p), m+1; \tanh^2 z\right) \tag{44}$$

$$\psi_{m,n}(z_1) = (\tanh z_1)^{\frac{1}{2}}(\cosh z_1)^{\mathrm{i}p} F\left(\tfrac{1}{2}(1+m-\mathrm{i}p), \tfrac{1}{2}(1-m-\mathrm{i}p), 1; \tanh^2 z_1\right). \tag{45}$$

Matching these solutions with the plane wave solution in the region $0 \ll z \ll K$ we obtain the quantization condition: $p_n = \frac{1}{4}\pi(2n-m+2)/(u+r_m)$ where $r_m = \psi(1) - \psi\left(\frac{1}{2}m + \frac{1}{2}\right)$, and $\psi(x)$ is the logarithmic derivative of the Γ function. This condition leads to the asymptotics

$$\frac{1}{6}\kappa_{m,n}(u) = m^2 + \pi^2\frac{(2n-m+2)^2}{16(u+r_m)^2} + O(1/u^5) \qquad n \geqslant \frac{1}{2}(m-1) \tag{46}$$

(see figure 4). We note that for $m \neq 0$ this UV behaviour is different from that for the sausage model eigenvalues, which is given by

$$\frac{1}{6}\kappa_{m,n}^{\mathrm{ssg}}(u) = m^2 + \pi^2\frac{(n+1)^2}{4(u+r_m)^2} + O(1/u^5) \qquad (n \geqslant 0). \tag{47}$$

The sausage model eigenvalue problem of equation (11) can also be written in the elliptic form of [11] with $k_s^2 = 1 - \exp(-4u)$:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\mathrm{cn}^2\left(2z|k_s^2\right)}{\mathrm{sn}^2\left(2z|k_s^2\right)} + \frac{m^2\,\mathrm{dn}^2\left(z|k_s^2\right)}{\mathrm{sn}^2\left(z|k_s^2\right)\mathrm{cn}^2\left(z|k_s^2\right)}\right)\psi_{m,n} = \frac{\kappa_{m,n}^{\mathrm{ssg}}k_s^2}{6}\psi_{m,n} \tag{48}$$

with boundary conditions $\psi_m \sim z^{m+1/2}$ at $z \to 0$; $\psi_m \sim (K - z)^{m+1/2}$ at $z \to K$. In the UV limit $u \to \infty$ the potential term in this equation at both ends tends to V given by equation (40). The normalizable solutions do not appear and both asymptotics can be described by the eigenfunction (44). The quantization condition for parameter p_n in this case leads to the asymptotics equation (47).

4. Numerical analysis

The matrix representation introduced in a previous section, while useful from the analytic viewpoint, is not the best choice if we want to compute the spectrum beyond perturbation theory. Actually an n-dimensional truncation of the matrix given in equation (22) is going to be essentially equivalent to nth order perturbation theory. We are now introducing another transformation of equation (13) which will allow us to efficiently explore the whole range $-\infty < u < \infty$.

We start from the fact that in the limit $u \to -\infty$ (and $m = 0$) there is a simple solution with

$$\Psi(y) = \sqrt{1 - \mathrm{e}^{-2(y+u)}} \equiv \sigma(y) \tag{49}$$

with $\kappa(u) \to 6$. Hence it seems promising to look for a solution of the form

$$\Psi(y) = \sigma(y)\psi(y). \tag{50}$$

To find the new differential equation it is convenient to write down the functional $\langle\mathcal{H}\rangle$ whose critical points are the eigenvalues: by denoting $\chi(y) \equiv 1 - \sinh 2u \sinh 2y$ and recalling $\rho(y)$ from equation (16), we have

$$\langle\mathcal{H}\rangle = \frac{\int_{-u}^{\infty}(\Psi(m^2 - \partial^2)\Psi - \chi\rho^{-2}\Psi^2)\,\mathrm{d}y}{\int_{-u}^{\infty}\rho^{-1}\Psi^2\,\mathrm{d}y}. \tag{51}$$

Now, by inserting equation (50), after an integration by parts, we find

$$\langle\mathcal{H}\rangle = \frac{\int_{-u}^{\infty}[\sigma^2\psi'^2 + (\sigma'^2 - \chi\sigma^2/\rho^2 - (\sigma\sigma')' + m^2\sigma^2)\psi^2]\,\mathrm{d}y}{\int_{-u}^{\infty}\sigma^2\rho^{-1}\psi^2\,\mathrm{d}y}. \tag{52}$$

The variational equation resulting from this functional

$$-\frac{\mathrm{d}}{\mathrm{d}y}\left(\sigma^2\frac{\mathrm{d}\psi}{\mathrm{d}y}\right) + V\psi = \frac{1}{6}\kappa\cosh 2u\,\sigma^2\rho^{-1}\psi \tag{53}$$

suggests introducing a new coordinate ξ such that $\mathrm{d}y = \sigma^2\,\mathrm{d}\xi$; we find

$$y = -u + \tfrac{1}{2}\log(1 + \mathrm{e}^{2\xi}) \tag{54}$$

and the equation simplifies to

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + \sigma^2 V(\xi, u)\psi = \frac{1}{6}\kappa(u)W(\xi, u)\psi(\xi) \tag{55}$$

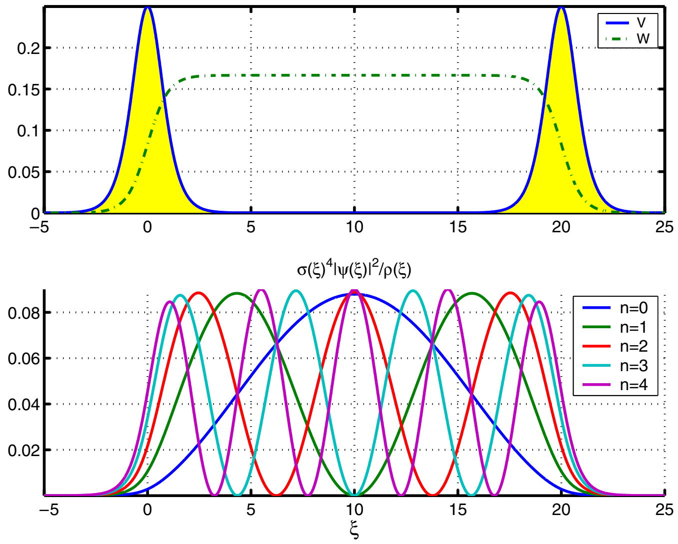

Figure 2. The generalized potentials V, W (top) and the first eigenfunctions ψ_n^2 for $u = 10$, $m = 0$.

with

$$\sigma^2 V = \frac{e^{2\xi}((1+e^{4u})^2 + (1+e^{2\xi})^2 - 1 + e^{4u}(e^{4\xi} + 3\,e^{2\xi}))}{(1+e^{2\xi})^2(1+e^{4u}+e^{2\xi})^2} + \frac{1}{4}m^2(1+\tanh\xi)^2 \tag{56}$$

$$W = \frac{e^{2\xi}(1+e^{4u})}{(1+e^{2\xi})(1+e^{4u}+e^{2\xi})}. \tag{57}$$

The crucial property of this new formulation is that it is now *regular on the whole real axis*. With appropriate boundary conditions (Neumann bc)[8] the eigenvalue equation can now be easily solved by standard sparse-matrix generalized eigenvalue routines[9]

See figures 2 and 3 for some typical waveforms. The density includes the measure appropriate for the new variable ξ, namely $\sigma(\xi)^4/\rho(\xi)$. In the limit $u \to -\infty$ we easily recover the discrete spectrum

$$\tfrac{1}{6}\kappa = \left\{(2j+1)^2 - m^2 | j = n + \tfrac{1}{2}m, n = 0, 1, 2, \ldots\right\} \tag{58}$$

as we already know from perturbation theory (see equation (33)). In the other limit, $u \gg 0$, the spectrum can be described as a monotonically decreasing flow towards $\kappa/6 = m^2$, *except for a finite number of eigenvalues* which have a value less than m^2: these peculiar 'bound states' are given by equation (43) and they are easily reproduced numerically (see figure 4). We easily check that they nicely agree with the asymptotic formulae already given in equation (47).

[8] It is somewhat tedious to trace the bc back from the original equation; suffice it to say that, intuitively speaking, since the constant solution is exact in the limit $u \to -\infty$, the Neumann bc are the natural ones.
[9] We successfully used the routine eigs in Matlab.

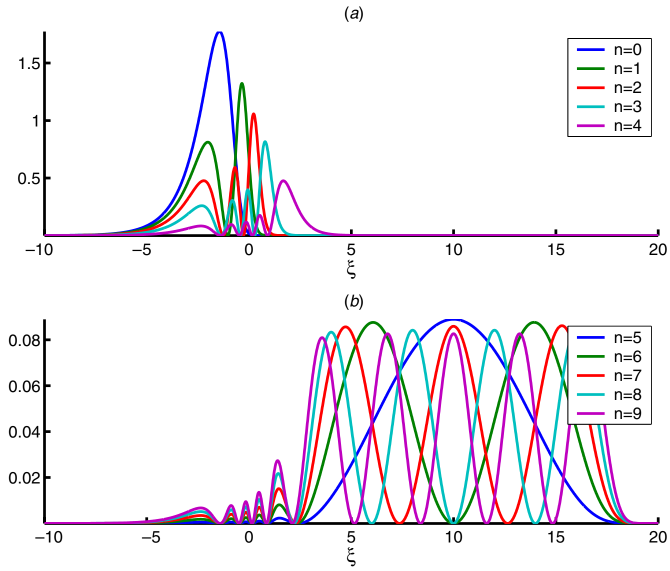

Figure 3. The densities at $u = 10, m = 10$: (*a*) 'bound states', (*b*) the 'continuum'.

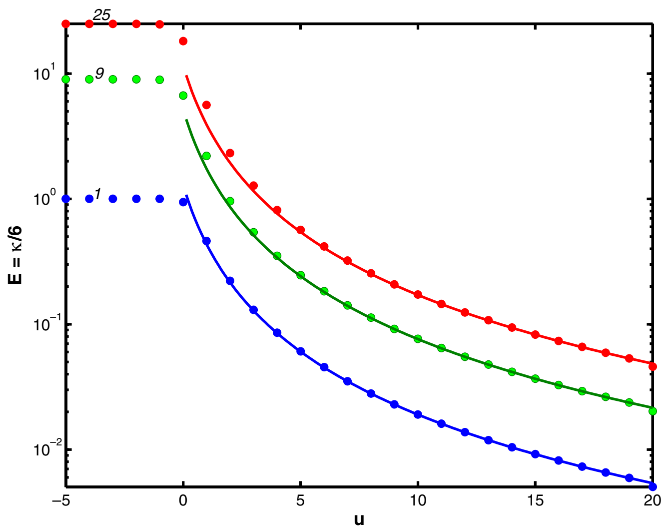

Figure 4. The asymptotic behaviour in the UV for the low lying states at $m = 0$: lines $\to$ equation (46), circles from numerical solution of equation (55).

5. Matching to TBA data

As already mentioned in the introduction for the special values of parameter

$$\frac{\nu}{4\pi} = \frac{1}{N} \qquad N = 3, 4, \ldots \tag{59}$$

the effective central charge of the SM (12) admits the exact (i.e. to all loops) calculation using the TBA equations. These equations can be derived from the factorized scattering theory

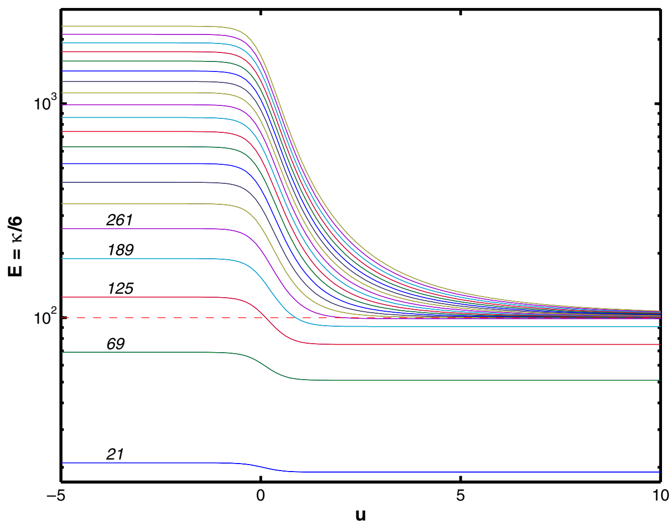

Figure 5. The spectrum flow for $m = 10$; the dashed line shows the continuum threshold.

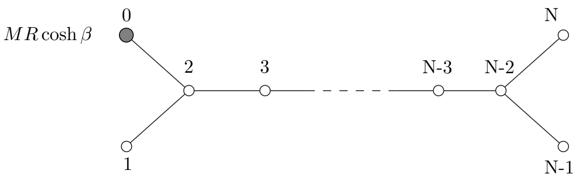

Figure 6. The extended Dynkin diagram for the sausage model.

of right (r) and left (l) moving massless particles which form the spectrum of our SM. The factorized scattering theory of massless particles is characterized by two-particle S-matrices $S_{rr}(\beta)$, $S_{ll}(\beta)$ and $S_{rl}(\beta)$ where β is the relative rapidity of scattering particles. We propose to discuss this scattering theory (as well as the perturbed CFT approach to our SM) in more detail in another publication. Here we only note that each of these three S-matrices coincides formally with the S-matrix for the massive particles in Z_N parafermionic CFT perturbed by the parafermionic operators. This S-matrix is described in detail in [29, 30]. This scattering theory results in TBA equations which form the common system of $N+1$ coupled non-linear integral equations for $N+1$ functions $\varepsilon_a(\beta)$ of rapidity variable $-\infty < \beta < \infty$. The TBA system has the form:

$$\rho_a(\beta) = \varepsilon_a + \frac{1}{2\pi}\int\sum_{b=0}^{N}\varphi_{ab}(\beta-\beta')\log(1+\exp(-\varepsilon_a(\beta')))\,\mathrm{d}\beta' \tag{60}$$

where $\varphi_{ab}(\beta) = \frac{1}{2\pi}\mathcal{I}_{ab}/\cosh\beta, \mathcal{I}$ being the incidence matrix of the extended affine D_N Dynkin diagram (see figure 7) and the source term

$$\rho_a(\beta) = \tfrac{1}{2}RM\ \exp(\beta)\delta_{a0} + \tfrac{1}{2}RM\ \exp(-\beta)\delta_{aN}. \tag{61}$$

The effective central charge can be calculated as

$$c(R) = \frac{3}{\pi^2}\int\sum_a \rho_a(\beta)\log(1+\exp(-\varepsilon_a(\beta))\,\mathrm{d}\beta. \tag{62}$$

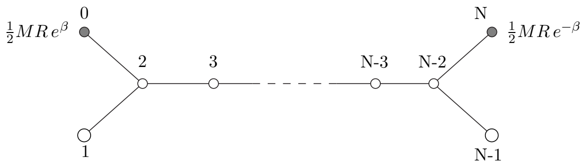

Figure 7. The extended Dynkin diagram for the SM (12).

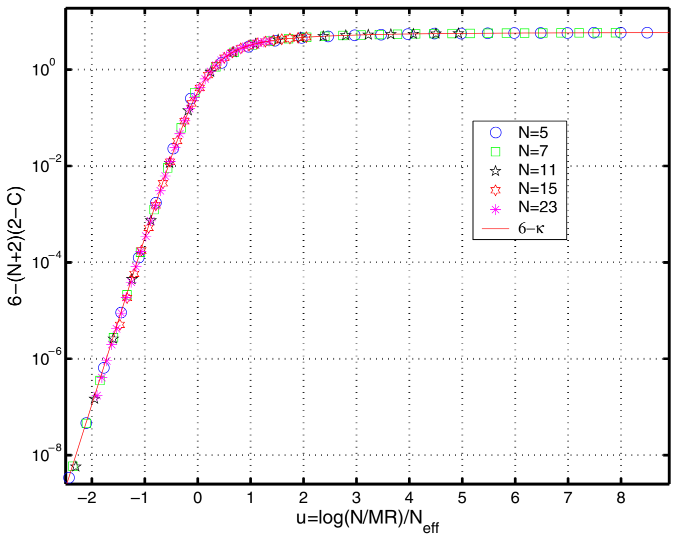

Figure 8. Matching κ to the TBA data.

The incidence matrix of these TBA equations is similar to that for the sausage model (see figure 6), the only difference coming from the source terms. For the massive sausage model (without topological term) $\rho_a(\beta) = RM\cosh(\beta)\delta_{a0}$.

The UV behaviour of $c(R)$ is determined only by the structure of the incidence matrix and we can find that in both cases the effective central charge approaches the limiting value $c_{\mathrm{UV}} = c(0) = 2$ logarithmically (see [11]) in agreement with equations (46) and (47). The analysis of TBA equations with source ρ_a given by equation (61) shows that in the IR limit $c_{\mathrm{IR}} = c(\infty) = 2 - 6/(N+2)$ coincides with the central charge of parafermionic CFT and the IR corrections to this value have a structure:

$$c(R, N) = 2 - \frac{6}{N+2} + b_2(N)\left(\frac{N+2}{MR}\right)^{8/(N+2)} + b_3(N)\left(\frac{N+2}{MR}\right)^{12/(N+2)} + \cdots. \tag{63}$$

The IR asymptotics of our theory can be described by the methods of perturbed CFT. This field theory is characterized by an integrable perturbative operator which belongs to the space of fields of the parafermionic CFT and has the dimension $\Delta_{\mathrm{pert}} = 1 + 2/(N+2)$. It gives us the possibility of calculating analytically the first corrections to expansion (63). The exact values for the coefficients $b_2(N)$ and $b_3(N)$ are presented in appendix C.

The effective central charge $c(R, N)$ was computed using the TBA equations numerically for several values of N ($N = 5, 7, 11, 15, 23, \ldots$). At large N the central charge is predicted to be given by

$$c(R, N) = 2 - \frac{\kappa_0(u)}{N+2} + O(1/N\log N). \tag{64}$$

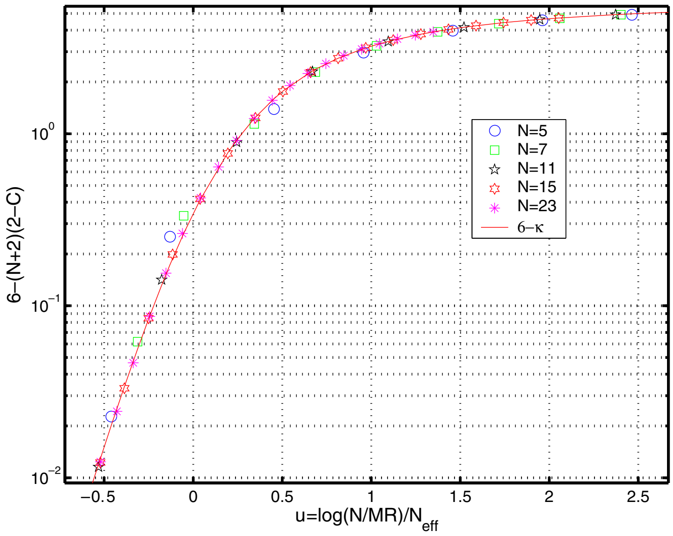

Figure 9. A close-up view of figure 8.

To verify this we have to relate u to the parameter MR entering the TBA equations. Within the one-loop approximation we have a freedom which can be used to fit the data in the best way. Asymptotically we expect $u \sim \log(N/MR)/N$, but finite N corrections are present and may be important. Empirically we find that a rather accurate choice is the following:

$$u = N_{\mathrm{eff}}^{-1} \log(N/MR) \tag{65}$$

$$N_{\mathrm{eff}} = \sqrt{(N+2)(N-2\tanh(4\log(N/MR)))} \tag{66}$$

which is used to build the plot of figure 8. It is quite evident that the data are increasingly well matched by κ as N increases (in the deep UV or IR the agreement is even better).

Finally let us note that the exact values of the coefficients $b_2(N)$ and $b_3(N)$, given in appendix C, have been reproduced with a high degree of accuracy by fitting the TBA data at various values of N.

6. Conclusions

We have found a unified treatment of the general spectral problem for the sausage model and its variant SM (12). The two formulations correspond to the same Heun equation defined on different domains in the complex plane and they are linked by a projective transformation. The spectral function $\kappa(u)$ can be studied both numerically by diagonalizing a discretized form of the differential operator, or analytically by a power series expansion around the IR point. The perturbative expansion can be pushed to high orders and it turns out to be convergent in the whole physical domain. Its asymptotic behaviour at high order is compatible with the leading UV behaviour, which has been computed analytically. In the ultraviolet regime, the spectrum, besides the expected continuous component, contains a set of bound states with angular momentum higher than one. The scaling functions for the ground state have been compared to the central charge computed via TBA equations: a very good agreement was found, adding good evidence for the interpretation of the SM (12) (at $\nu = 4\pi/N$) as the field theory describing the RG flow to the Z_N parafermionic CFT.

Acknowledgments

EO would like to thank R K Ellis and his colleagues of the Theory Group at Fermilab for the kind hospitality he enjoyed while this work was done. We warmly thank R De Pietri who developed the symbolic code. This work was supported in part by the EU under contract HPRN-CT-2002-00325 and grant INTAS-OPEN-00-00055.

Appendix A

We recall some standard results from Rayleigh–Schrödinger perturbation theory for non-degenerate levels, which should be found in any textbook on quantum mechanics. Stationary perturbation theory is most efficiently formulated as an iterative algorithm. Let $H = H_0 + \varepsilon V$, E_0 any one of the unperturbed eigenvalues, with eigenstate $|E_0\rangle$. Then the perturbed eigenvalue $E(\varepsilon)$ can be expanded as

$$E(\varepsilon) = E_0 + \sum_{n\geqslant 1} \delta_n \varepsilon^n \tag{A1}$$

while the eigenstate is given by a vector series in terms of auxiliary vectors $|\eta_n\rangle$ as

$$|E(\varepsilon)\rangle = |E_0\rangle + \sum_{n\geqslant 1} \varepsilon^n |\eta_n\rangle. \tag{A2}$$

The expansion coefficients can be computed through the following recursive algorithm:
Let

$$|\eta_0\rangle = |E_0\rangle$$

and

$$\mathcal{R}_0 = \frac{1 - |E_0\rangle\langle E_0|}{H_0 - E_0}$$

then for any $n \geqslant 1$ we have

$$\begin{cases} \delta_n = \langle E_0|V|\eta_{n-1}\rangle \\ |\eta_n\rangle = \mathcal{R}_0 \left\{-V|\eta_{n-1}\rangle + \sum_{k=1}^{n} \delta_k |\eta_{n-k}\rangle\right\}. \end{cases} \tag{A3}$$

The algorithm can be easily translated into any symbolic manipulation language. In principle we should take care of defining the matrix V in an orthonormal basis; this would introduce some square roots in our matrix, equation (22), while at the end the coefficients turn out to be rational. Actually the following lemma tells us that we may comfortably work with the unnormalized basis.

Lemma. *Let H, H_0, V, $|E_0\rangle$, $E(\varepsilon)$, $\mathcal{R}$, δ_n, $|\eta_n\rangle$ be defined as above. Let us assume further, to avoid any convergence problem, that V is a finite-band matrix along the main diagonal. Let S be any non-singular Hermitian operator commuting with $\mathcal{H}_0$. Then we may substitute V with SVS^{-1} in the iterative algorithm (A3), leaving everything else unchanged.*

Proof. Let δ'_n, $|\eta'_n\rangle$ denote the sequence constructed by inserting SVS^{-1} into equation (A3). Since S commutes with H_0, we must have $S|E_0\rangle = s_0|E_0\rangle$ for some non-zero s_0. Hence

$$\delta'_n = \langle E_0|V\, s_0 S^{-1}|\eta'_{n-1}\rangle. \tag{A4}$$

Let us define $|\eta''_n\rangle = s_0 S^{-1}|\eta'_{n-1}\rangle$. It is easy to check that the sequence $\{\delta'_n, |\eta''_n\rangle\}$ satisfies the same recursion as $\{\delta_n, |\eta_n\rangle\}$, and moreover $|\eta''_0\rangle = s_0 S^{-1}|E_0\rangle = |E_0\rangle$, hence the two sequences must coincide. □

Note. The statement in the lemma is strictly perturbative. H and SHS^{-1} could be inequivalent as operators, since S and/or S^{-1} may be unbounded, still they share the same perturbative expansion.

Appendix B

We report the explicit expression of the series expansion of $\kappa_{m,n}$ for small values of (n, m). Of course this is just a sample; the code can generate them to any order, the only limitation being the computer's physical memory and time.

$$\kappa_{0,0} = 6 - \lambda^2 - \frac{1}{2}\lambda^3 - \frac{229}{720}\lambda^4 - \frac{109}{480}\lambda^5 - \frac{62\,999}{362\,880}\lambda^6 - \frac{20\,159}{145\,152}\lambda^7 - \frac{299\,803\,787}{2612\,736\,000}\lambda^8 - \frac{72\,503\,387}{746\,496\,000}\lambda^9 - \frac{173\,336\,436\,487}{2069\,286\,912\,000}\lambda^{10} + O(\lambda^{11})$$

$$\kappa_{1,0} = 18 - 4\lambda - \frac{16}{9}\lambda^2 - \frac{352}{405}\lambda^3 - \frac{1972}{3645}\lambda^4 - \frac{17\,408}{45\,927}\lambda^5 - \frac{701\,314}{2460\,375}\lambda^6 - \frac{34\,835\,788}{155\,003\,625}\lambda^7 - \frac{204\,567\,413}{1116\,026\,100}\lambda^8 - \frac{1588\,447\,666\,493}{10\,358\,117\,240\,625}\lambda^9 - \frac{4782\,354\,354\,298\,021}{36\,543\,437\,624\,925\,000}\lambda^{10} + O(\lambda^{11})$$

$$\kappa_{0,1} = 54 - 24\lambda - \frac{27}{5}\lambda^2 - \frac{27}{10}\lambda^3 - \frac{23\,949}{14\,000}\lambda^4 - \frac{34\,047}{28\,000}\lambda^5 - \frac{370\,287}{400\,000}\lambda^6 - \frac{826\,209}{1120\,000}\lambda^7 - \frac{146\,655\,243\,891}{241\,472\,000\,000}\lambda^8 - \frac{35\,351\,959\,491}{68\,992\,000\,000}\lambda^9 - \frac{197\,594\,782\,006\,203}{448\,448\,000\,000\,000}\lambda^{10} + O(\lambda^{11})$$

$$\kappa_{1,1} = 90 - \frac{196}{5}\lambda - \frac{9664}{1125}\lambda^2 - \frac{7627\,904}{1771\,875}\lambda^3 - \frac{217\,386\,688}{79\,734\,375}\lambda^4 - \frac{173\,655\,964\,928}{89\,701\,171\,875}\lambda^5 - \frac{485\,256\,409\,132\,928}{329\,651\,806\,640\,625}\lambda^6 - \frac{955\,858\,372\,577\,612\,032}{815\,888\,221\,435\,546\,875}\lambda^7 - \frac{176\,847\,414\,696\,606\,187\,696}{183\,574\,849\,822\,998\,046\,875}\lambda^8 - \frac{12\,815\,580\,494\,902\,423\,265\,411\,456}{15\,786\,519\,210\,528\,717\,041\,015\,625}\lambda^9 + O(\lambda^{10}).$$

Appendix C

In this appendix we give the exact values for the first IR correction to the levels $e_{mj}(R)$ and two first corrections to the effective central charge or to the ground state energy $e_0(R)$. We express these corrections in terms of parameter M entering the TBA equations. These corrections can be calculated analytically using the methods of integrable perturbed CFT. We consider the case $\nu/4\pi = 1/N$. In this case the IR limit of SM (12) is described by parafermionic CFT. The conformal dimensions of the primary fields in this CFT are characterized by two quantum numbers m and j ($j = |m/2|, |m/2| + 1 \cdots \leqslant N/2$) and have the form:

$$\Delta_{mj} = \frac{j(j+1)}{N+2} - \frac{m^2}{4N}. \tag{C1}$$

It is convenient to introduce $D_{mj} = (N+2)\Delta_{mj}$. As was noted in the introduction, the limiting values of $(e_{mj} - e_0)/24$ should coincide with Δ_{mj}. With the first IR correction these values are

$$\frac{(N+2)(e_{mj}(R) - e_0(R))}{6} = 4D_{mj} - D_{mj}^2 \frac{b_1(j,N)}{j(j+1)} \left(\frac{N+2}{MR}\right)^{4/(N+2)} + \cdots \tag{C2}$$

where the coefficient $b_1(j,N)$ can be expressed through the function $g(x) = \Gamma(1+x)/\Gamma(1-x)$ and has a form

$$b_1(j,N) = \frac{2N^2}{(N+2)^2} \frac{g(1/(N+2))^2 g((2j+2)/(N+2))}{g(2/(N+2)) g(2j/(N+2))} (8\pi)^{4/(N+2)}. \tag{C3}$$

We note that in the large N limit the left-hand side of equation (C2) tends to $(\kappa_{mj} - \kappa_0)/6$. For $4D_{mj}$ and $b_1(j,N)$ we have

$$4D_{mj} = (4j(j+1) - m^2)(1 + O(1/N)) \qquad b_1(j,N) = 2 + O(1/N) \tag{C4}$$

if we now define the parameter u by the relation: $[(N+2)/MR]^{4/(N+2)} = \exp(4u) = \lambda/(1-\lambda)$ we can see that the first term in the expansion given in equation (33) coincides with equation (C2) at one-loop accuracy.

For the effective central charge $c(R,N)$ (or for the ground state level $e_0(R)$) it is possible to calculate analytically two further IR corrections. Namely, the coefficients b_2 and b_3 in expansion (63) are given by

$$(N+2)b_2(N) = \frac{N^2(N-2)^2 g(1/(N+2)) g(3/(N+2)) (8\pi)^{8/(N+2)}}{(N+4)^2(N+6)^2 g(4/(N+2)) g(-2/(N+2))^2} = 1 + O(1/N) \tag{C5}$$

$$(N+2)b_3(N) = -\frac{3N^4(N-4)^2 g(2/(N+2)) g(4/(N+2)) (8\pi)^{12/(N+2)}}{2(N+4)^4(N+8)^2 g(6/(N+2)) g(-3/(N+2))^2} = -\frac{3}{2} + O(1/N). \tag{C6}$$

It is easy to see from these equations that with one-loop accuracy the IR expansion for the function $\kappa_0(u) = 6 - \lambda^2 - \lambda^3/2 + \cdots$ coincides with the exact IR expansion for the function $(N+2)(2 - c(R,N))$.

In the UV limit the leading term for the levels $e_{mj}(R)$ (for $j \geqslant m - 1/2$) can also be calculated exactly and has a form:

$$Ne_{mj}(R) = 6m^2 + \frac{3(j-m+1)^2\pi^2 N(N-2)}{2Z_m^2(R)} + O\left(1/Z_m^5\right) \tag{C7}$$

where $Z_m(R) = \log(8\pi(N-2)/RM) + (N-2)(\psi(1) - \psi((m+1)/2)) + \psi(1)$. This asymptotic behaviour coincides at one-loop accuracy with equation (46), where $n = j - m/2$.

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